Information-theoretical properties of the d-dimensional blackbody radiation

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Space dimensionality effects

For most purposes we take for granted that we live in a three-dimensional universe.

However, the dimensionality of our space

- is not derived from any physical law, but
- we perceive that we can move ourselves in three distinct directions.

Perception of dimensionality is obviously conditioned by the limitations of our senses, and a possible world with more than three dimensions would be properly detected by means of

- experiments realized in laboratories (e.g., LHC experiments)
- cosmological observations (e.g., COBE, PLANCK mission)

There exists a growing interest in the physics of multidimensional universes, because

- space dimensionality is well known to modify the physical solutions of the quantum wave equations of the systems, and thus all their properties,

- many quantum systems and phenomena possess natural generalizations in which the number of degrees of freedom is a free parameter,

- the dimensionality of a quantum system can be used as a technological resource, as recently shown in quantum computation.

  A sufficiently high dimensionality of entanglement is necessary for any quantum speed up.

- ......

General purpose

To study the dimensionality effects on the Black Body Radiation (BBR) in an universe with arbitrary dimensionality.

A blackbody is defined as a body with a rich energy spectrum, capable of exciting all frequencies of light by thermalization. So that,

all blackbodies at the same temperature emit
thermal radiation with the same spectrum.

To quantify the degree of *order/disorder* and *simplicity/complexity* of the d-dimensional BlackBody Radiation (BBR).

The concepts *order/disorder* and *simplicity/complexity* for a complex system are not at all simple: *they need several measures to capture our intuitive notions in the appropriate manner.*

Specific aim

To quantify how simple or how complex is the d-dimensional BBR by means of the entropy and complexity measures of its associated Planck density.

These quantities

- are not measures of extrinsic type (algorithmic, computational,...), closely related to the time required for a computer to solve a given problem, BUT
- are density-dependent quantities; so, intrinsic properties of the blackbody, closely related to the main *macroscopic features* of its associated probability density (e.g., irregularities, extent, fluctuations, smoothing,...).

Planck frequency density of the d-dimensional BlackBody

The fundamental laws in spaces with non-standard dimensionalities can be generalized from what we know in our three-dimensional space using thermodynamics and statistical mechanics arguments.

For our purposes, the spectral energy density of a d-dimensional $(d > 1)$ blackbody at temperature T (i.e., the energy per frequency and volume units contained in the frequency interval $(\nu,\, \nu + d\nu)$ inside a d-dimensional enclosure maintained at temperature T) is given by the generalized Planck radiation law

$$\rho_T^{(d)}(\nu) = \frac{2(d-1)h\left(\frac{\sqrt{\pi}}{c}\right)^d}{\Gamma\left(\frac{d}{2}\right)} \frac{\nu^d}{e^{\frac{h\nu}{k_B T}} - 1}, \tag{1}$$

where h and k_B are the Planck and Boltzmann constants, respectively.

Spreading/entropy measures of a probability density $\rho(x)$

Ordinary moments

$$\langle x^k \rangle = \int x^k \rho(x)\, dx\,;\ k = 0, 1, 2, \ldots$$

$\Longrightarrow$ Variance $V[\rho] = \langle x^2 \rangle - \langle x \rangle^2$

Disequilibrium

$$\mathcal{D}[\rho] = \int [\rho(x)]^2\, dx$$

Shannon entropy

$$S[\rho] = \int \rho(x) \log \rho(x) dx$$

Fisher information

$$F[\rho] = \int \frac{|\rho'(x)|^2}{\rho(x)} dx$$

Spreading/entropy measures of $\rho_T^{(d)}(\nu), d > 1$.

The main quantifiers of the frequency spreading of the spectral density of a d-dimensional blackbody, $\rho_T^{(d)}(\nu), d > 1$, are:

Variance

$$V(d,T) = \langle \nu^2 \rangle - \langle \nu \rangle^2 \; = \; C_1(d) \left(\frac{k_B T}{h} \right)^2,$$

Disequilibrium

$$D(d,T) = \int_0^\infty \rho(\nu)^2 \, d\nu \; = \; C_2(d) \frac{h}{k_B T},$$

Shannon entropy

$$S(d,T) = -\int_0^\infty \rho(\nu) \log \rho(\nu) \, d\nu \; = \; -\log \frac{h}{k_B T} + C_3(d)$$

Fisher information

$$F(d,T) = \int_0^\infty \frac{[\rho'(\nu)]^2}{\rho(\nu)} \, d\nu \; = \; C_4(d) \left(\frac{h}{k_B T} \right)^2$$

Characteristic frequencies of the d-dimensional BBR

Predominant frequency (generalized Wien's displacement law)

$$\nu_{max}(d,T) \;\; = \;\; C_0(d)\,\frac{k_B T}{h},$$

Spreading/entropy-based frequencies:

$$\nu_{Heis}(d,T) \;\; = \;\; \sqrt{V(d,T)} \; = \; \sqrt{C_1(d)}\,\frac{k_B T}{h},$$

$$\nu_{Shan}(d,T) \;\; = \;\; \exp\left(S(d,T)\right) \; = \; \exp\left(C_3(d)\right)\frac{k_B T}{h},$$

$$\nu_{Fish}(d,T) \;\; = \;\; \frac{1}{\sqrt{F(d,T)}} \; = \; \frac{1}{\sqrt{C_4(d)}}\,\frac{k_B T}{h},$$

Note that

- they depend linearly on the temperature, and
- the C_i-constants only depend on the universe dimensionality.

Now, for convenience, we define the dimensionless quantity

$$x_i = \frac{h}{k_B T}\nu_i, \quad \text{for i = max, Heis, Shan, Fish,}$$

Then, we obtain the following relative comparison among the four Wien-like laws:

$$x_{Fish} < x_{max} < x_{Heis} < x_{Shan} \quad , \quad d = 2 \tag{2}$$

$$x_{Fish} < x_{Heis} < x_{max} < x_{Shan} \quad , \quad 3 \leq d \leq 17 \tag{3}$$

$$x_{Fish} < x_{Heis} < x_{Shan} < x_{max} \quad , \quad d \geq 18 \tag{4}$$

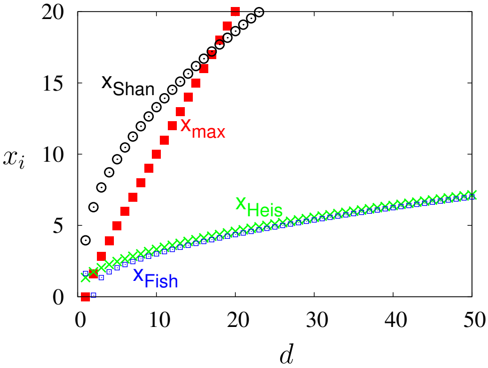


Figure: Wien-like laws of the four characteristic frequencies $x_i(d)$ for the d-dimensional blackbody radiation

Note that this behaviour is universally valid in the sense that it holds for any value of the absolute temperature T.

Case $d = 3$

For the three-dimensional BB at temperature T, we find:

$$\nu_{max}(3,T) \simeq 2.82144 \, \frac{k_B T}{h}, \tag{5}$$

$$\nu_{Heis}(3,T) \simeq 2.02812 \, \frac{k_B T}{h}, \tag{6}$$

$$\nu_{Shan}(3,T) \simeq 7.66411 \frac{k_B T}{h}, \tag{7}$$

$$\nu_{Fish}(3,T) \simeq 1.34193 \, \frac{k_B T}{h}, \tag{8}$$

Case $d = 3$ and $T = 2.725$K: Cosmic Microwave Background (CMB)

The CMB radiation today is known to be the most perfect blackbody radiation ever observed in nature, with a temperature of about 2.725 K.

In addition to the known predominant frequency located at

$$\nu_{max}(cmb) = 1.60201 \cdot 10^{11}\text{Hz} \tag{9}$$

we've found the following three novel entropy-based characteristic frequencies:

$$\nu_{Fish}(cmb) \simeq 6.57748 \cdot 10^{10}\,\text{Hz}, \tag{10}$$

$$\nu_{Heis}(cmb) \simeq 1.15156 \cdot 10^{11}\,\text{Hz}, \tag{11}$$

$$\nu_{Shan}(cmb) \simeq 4.35167 \cdot 10^{11}\,\text{Hz}. \tag{12}$$

These three entropy-based characteristic frequencies of the CMB spectrum have not yet been experimentally checked.

Conjecture:

The Fisher-information-based frequency ν_{Fish} is an appropriate quantifier of the anisotropies of the CMB radiation.

These anisotropies are still very controversial, despite of so many efforts.

- Their physical origin is not yet clear (interaction between photons and other microscopic particles and/or to the nonextensive statistics environment which is presumably associated to the long-range interactions,...)
- They are being theoretically determined in the framework of some nonlinear models and non-extensive theories of CMB.
- The Planck Mission is providing a great deal of data about them which are being presently interpreted.

Complexity measures of a probability density

Fisher-Shannon complexity

$$\mathcal{C}_{\mathsf{FS}}[\rho] = F[\rho] \times \frac{1}{2\pi e} \exp\left(2\, S[\rho]\right)$$

It measures the gradient content of ρ together with its spreading.

LMC complexity

$$\mathcal{C}_{\mathsf{LMC}}[\rho] = \mathcal{D}[\rho] \times \exp\left(S[\rho]\right)$$

It measures the combined balance of the average height of the density and its effective extent.

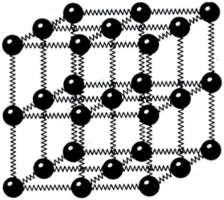
Crámer-Rao complexity

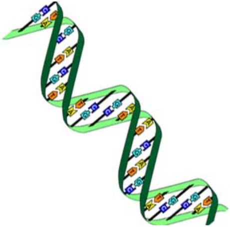
$$\mathcal{C}_{\mathsf{CR}}[\rho] = F[\rho] \times V[\rho]$$

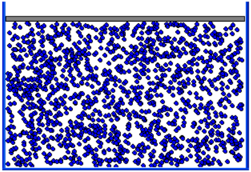
It quantifies the gradient content of the density jointly with the frequency concentration around its mean value.

On the other hand, a measure of complexity tell us how easily a model can be modelled.

Indeed, when $d = 3$,







Note that:

- a *completely ordered system* (e.g., a perfect cristal) as well as a *totally disordered one* (e.g., an isolated ideal gas), are not complex systems.
- Between these two extreme cases, we find many others in which order and disorder are involved simultaneously.

Complexity measures of the d-dimensional BBR

From the previous entropic findings one can calculate the Fisher-Shannon, LMC and Crámer-Rao complexity measures of the d-dimensional Planck frequency density as

$$\begin{aligned} \mathcal{C}_{FS}(d,T) &= F(d,T)\cdot\frac{1}{2\pi\, e}e^{2\,S(d,T)} = C_4(d)\frac{1}{2\pi\, e}e^{2C_3(d)}, \\ \mathcal{C}_{LMC}(d,T) &= D(d,T)\; e^{S(d,T)} = C_2(d)\, e^{C_3(d)}, \\ \mathcal{C}_{CR}(V,T) &= F(d,T)\cdot V(d,T) = C_4(d)\, C_1(d), \end{aligned}$$

respectively, where all the dimensionless constants C_i, with i $=$ 1, 2, 3 and 4, are explicitly known.

Interestingly, we notice that

- all three complexities of the d-dimensional blackbody radiation depend neither on temperature nor on the Planck and Boltzmann constants, but they only depend on dimensionality.
- This indicates that these three measures of complexity have an universal character for any d-dimensional blackbody.
- The dimensionality dependence of the complexity measures is shown in Figure 2.

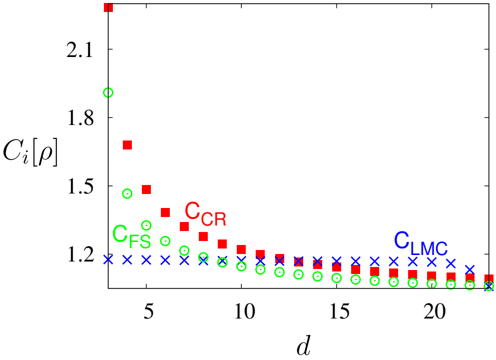


Figure: The Crámer-Rao, Fisher-Shannon and LMC complexity measures $\mathcal{C}[\rho]$ of the d-dimensional blackbody radiation in terms of d.

Interestingly, we observe that:

- The three measures of complexity monotonically decrease with different slopes when d is increasing.

- The LMC complexity is practically constant. The other two measures of complexity clearly decrease, mainly because of the Fisher component included in both quantities.

- The Fisher-Shannon measure decreases faster than the Crámer-Rao one, basically because the Shannon component included in the former measure is bigger than the variance component of the latter complexity.

- Conjecture: the Fisher-Shannon and Crámer-Rao measures of complexity can be used as quantifiers of the space dimensionality of the universe as grasped by the blackbody spectrum.

- Moreover, they drastically vary even with small dimensional fluctuations of the spectrum because of the huge variations of their Fisher ingredient, as already mentioned. From this point of view, these two measures of complexity could be eventually used to detect dimensional anisotropies of the spectrum.

- Finally note that for $d = 3$, one has

$$\begin{aligned}
\mathcal{C}_{FS}(d,T) &= 1.90979, \\
\mathcal{C}_{LMC}(d,T) &= 1.17685, \\
\mathcal{C}_{CR}(V,T) &= 2.28415,
\end{aligned}$$

the 3-dimensional blackbody radiation.

Conclusions

We have calculated the dimensionality effects on the entropy and complexity measures of the BlackBody Radiation (BBR).

Entropy analysis of the d-dimensional BBR:

- We have found that three new characteristic entropy-based frequencies in the spectrum of the d-dimensional BBR.

  They obey a Wien-like law.

- The values of these entropy-based frequencies for the CMB radiation are predicted.

  It is conjectured the potential interest of the Fisher-information-based frequency ν_{Fish} to quantify the CMB anisotropies.

Complexity analysis of the d-dimensional BBR:

- We have shown that the three main measures of complexity do not depend on the temperature, but only on the universe dimensionality.

  That is, they are universal constants.

- For the (three-dimensional) CMB which baths our universe, the values of three complexity measures are predicted.

  We have explicitly obtained that the Crámer-Rao complexity is bigger than the Fisher-Shannon and LMC quantities. This is mainly because of the high smoothness and extent of the corresponding 3-dimensional Planck density.